MODERATE DEVIATIONS FOR THE EIGENVALUE COUNTING FUNCTION OF WIGNER MATRICES

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Abstract: We establish a moderate deviation principle (MDP) for the number of eigenvalues of a Wigner matrix in an interval. The proof relies on fine asymptotics of the variance of the eigenvalue counting function of GUE matrices due to Gustavsson. The extension to certain families of Wigner matrices is based on the Tao and Vu Four Moment Theorem and applies localization results by Erdös, Yau and Yin. Moreover we investigate families of covariance matrices as well.

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1. Introduction

Recently, in [4] the Central Limit Theorem (CLT) for the eigenvalue counting function of Wigner matrices, that is the number of eigenvalues falling in an interval, was established. This *universality result* relies on fine asymptotics of the variance of the eigenvalue counting function, on the Fourth Moment Theorem due to Tao and Vu as well as on recent localization results due to Erdös, Yau and Yin. See also [3]. Our paper is concerned with the moderate deviation principle (MDP) of the eigenvalue counting function. We will start with the MDP for Wigner matrices where the entries are Gaussian (the so-called Gaussian unitary ensemble (GUE)), first proven by the authors in [11]. Next we establish a MDP for individual eigenvalues in the bulk of the semicircle law (which is an MDP corresponding to the Gaussian behaviour proved in [14]). This MDP will be extended to certain families of Wigner matrices by means of the Four Moment Theorem (see [21], [20]). It seem to be the first application of the Four Moment Theorem to be able to obtain not only universality of convergence in distribution but also to obtain deviations results on a logarithmic scale, universally. Finally a strategy based on the MDP for individual eigenvalues in the bulk will be shown to imply the MDP for the eigenvalue counting function, universally for certain Wigner matrices. In the meantime, we successfully apply the Four Moment Theorem to obtain MDPs also at the edge of the spectrum as well as for the determinant of certain Wigner matrices, see [9], [10].

Consider two independent families of i.i.d. random variables $(Z_{i,j})_{1\leq i<j}$ (complex-valued) and $(Y_i)_{1\leq i}$ (real-valued), zero mean, such that $\mathbb{E}Z_{1,2}^2 = 0, \mathbb{E}|Z_{1,2}|^2 = 1$ and $\mathbb{E}Y_1^2 = 1$. Consider the (Hermitian) $n \times n$ matrix M_n with entries $M_n^*(j,i) = M_n(i,j) = Z_{i,j}/\sqrt{n}$ for $i < j$ and $M_n^*(i,i) = M_n(i,i) = Y_i/\sqrt{n}$. Such a matrix is called *Hermitian Wigner matrix*. An important example of Wigner matrices is the case where the entries are Gaussian, giving rise to the so-called Gaussian Unitary Ensembles (GUE). GUE matrices will be denoted by M_n'. In this case, the joint law of the eigenvalues is known, allowing a good description of their limiting behaviour both in the global and local regimes (see [1]). In the Gaussian case, the distribution of the matrix is invariant by the action of the group $SU(n)$. The eigenvalues of the matrix M_n' are independent of the eigenvectors which are Haar distributed. If $(Z_{i,j})_{1\leq i<j}$ are real-valued the *symmetric Wigner matrix* is defined analogously and the case of Gaussian variables with $\mathbb{E}Y_1^2 = 2$ is of particular importance, since their law is invariant under the action of the orthogonal group $SO(n)$, known as Gaussian Orthogonal Ensembles (GOE).

Denote by $\lambda_1, \ldots, \lambda_n$ the real eigenvalues of the normalised Hermitian (or symmetric) Wigner matrix $W_n = \frac{1}{\sqrt{n}} M_n$. The Wigner theorem states that the empirical measure

$$L_n := \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$$

on the eigenvalues of W_n converges weakly almost surely as $n \to \infty$ to the semicircle law

$$d\varrho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\, 1_{[-2,2]}(x)\, dx,$$

(see [1, Theorem 2.1.21, Theorem 2.2.1]). Consequently, for any interval $I \subset \mathbb{R}$,

$$\frac{1}{n} N_I(W_n) := \frac{1}{n}\sum_{i=1}^n 1_{\{\lambda_i \in I\}} \to \varrho_{sc}(I)$$

almost surely as $n \to \infty$. At the fluctuation level, it is well known that for the GUE, $W_n' := \frac{1}{\sqrt{n}} M_n'$ satisfies a CLT (see [18]): Let I_n be an interval in $\mathbb{R}$. If $\mathbb{V}(N_{I_n}(W_n')) \to \infty$ as $n \to \infty$, then

$$\frac{N_{I_n}(W_n') - \mathbb{E}[N_{I_n}(W_n')]}{\sqrt{\mathbb{V}(N_{I_n}(W_n'))}} \to N(0,1)$$

as $n \to \infty$ in distribution.

In [14] the asymptotic behavior of the expectation and the variance of the counting function $N_{I_n}(W_n')$ for intervals $I_n = [y(n), \infty)$ with $y(n) = G^{-1}(k/n)$ (where $k = k(n)$ is such that $k/n \to a \in (0,1)$ – *strictly in the bulk*–, and G denotes the distribution function of the semicircle law) was established:

$$\mathbb{E}[N_{I_n}(W_n')] = n - k(n) + O\big(\frac{\log n}{n}\big) \text{ and } \mathbb{V}(N_{I_n}(W_n')) = \big(\frac{1}{2\pi^2} + o(1)\big)\log n. \tag{1.1}$$

The proof applied strong asymptotics for orthogonal polynomials with respect to exponential weights, see [5]. In particular the CLT holds for $N_I(W_n')$ if $I = [y, \infty)$ with $y \in (-2,2)$, and moreover in this case one obtains

$$\frac{N_I(W_n') - n\varrho_{sc}(I)}{\sqrt{\frac{1}{2\pi^2}\log n}} \to N(0,1)$$

as $n \to \infty$ (called the CLT with numerics). These conclusions were extended to non-Gaussian Wigner matrices in [4].

Certain deviations results and concentration properties for Wigner matrices were considered. Our aim is to establish certain moderate deviation principles. Recall that a sequence of laws $(P_n)_{n\ge 0}$ on a Polish space Σ satisfies a large deviation principle (LDP) with good rate function $I : \Sigma \to \mathbb{R}_+$ and speed s_n going to infinity with n if and only if the level sets $\{x : I(x) \le M\}$, $0 \le M < \infty$, of I are compact and for all closed sets F

$$\limsup_{n\to\infty} s_n^{-1} \log P_n(F) \le -\inf_{x\in F} I(x)$$

whereas for all open sets O

$$\liminf_{n\to\infty} s_n^{-1} \log P_n(O) \ge -\inf_{x\in O} I(x).$$

We say that a sequence of random variables satisfies the LDP when the sequence of measures induced by these variables satisfies the LDP. Formally a moderate deviation principle is nothing else but the LDP. However, we speak about a moderate deviation principle (MDP) for a

sequence of random variables, whenever the scaling of the corresponding random variables is between that of an ordinary Law of Large Numbers (LLN) and that of a CLT.

Large deviation results for the empirical measures of Wigner matrices are still only known for the Gaussian ensembles since their proof is based on the explicit joint law of the eigenvalues, see [2] and [1]. A moderate deviation principle for the empirical measure of the GUE or GOE is also known, see [6]. This moderate deviations result does not have yet a fully universal version for Wigner matrices. It has been generalised to Gaussian divisible matrices with a deterministic self-adjoint matrix added with converging empirical measure [6] and to Bernoulli matrices [8].

Our first result is a MDP for the number of eigenvalues of a GUE matrix in an interval. It is a little modification of [11, Theorem 5.2]. In the following, for two sequences of real numbers $(a_n)_n$ and $(b_n)_n$ we denote by $a_n \ll b_n$ the convergence $\lim_{n\to\infty} a_n/b_n = 0$.

Theorem 1.1. *Let M_n' be a GUE matrix and $W_n' := \frac{1}{\sqrt{n}} M_n'$. Let I_n be an interval in $\mathbb{R}$. If $\mathbb{V}(N_{I_n}(W_n')) \to \infty$ for $n \to \infty$, then for any sequence $(a_n)_n$ of real numbers such that*

$$1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W_n'))} \tag{1.2}$$

the sequence $(Z_n)_n$ with

$$Z_n = \frac{N_{I_n}(W_n') - \mathbb{E}[N_{I_n}(W_n')]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n'))}}$$

satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$.

Remark 1.2. Let $I = [y, \infty)$ with $y \in (-2, 2)$. An easy consequence of Theorem 1.1 is that the sequence $(\hat{Z}_n)_n$ with

$$\hat{Z}_n = \frac{N_I(W_n') - n\varrho_{sc}(I)}{a_n \sqrt{\frac{1}{2\pi^2} \log n}}$$

satisfies the MDP with the same speed, the same rate function, and in the same regime (1.2) (called the MDP with numerics).

In our paper we extend these conclusions to certain non-Gaussian Hermitian Wigner matrices.

Tail-condition (T)**:** Say that M_n satisfies the tail-condition (T) if the real part η and the imaginary part $\overline{\eta}$ of M_n are independent and have a so-called stretched exponential decay: there are two constants C and C' such that

$$P\big(|\eta| \geq t^C\big) \leq e^{-t} \text{ and } P\big(|\overline{\eta}| \geq t^C\big) \leq e^{-t}$$

for all $t \geq C'$.

We say that two complex random variables η_1 and η_2 *match to order k* if

$$\mathbb{E}\big[\mathrm{Re}(\eta_1)^m \,\mathrm{Im}(\eta_1)^l\big] = \mathbb{E}\big[\mathrm{Re}(\eta_2)^m \,\mathrm{Im}(\eta_2)^l\big]$$

for all $m, l \geq 0$ such that $m + l \leq k$.

The following theorem is the main result of our paper:

Theorem 1.3. *Let M_n be a Hermitian Wigner matrix whose entries satisfy tail-condition* (T) *and match the corresponding entries of GUE up to order 4. Set $W_n := \frac{1}{\sqrt{n}} M_n$. Then, for any $y \in (-2,2)$ and $I(y) = [y, \infty)$, with $Y_n := N_{I(y)}(W_n)$, for any sequence $(a_n)_n$ of real numbers such that*

$$1 \ll a_n \ll \sqrt{\mathbb{V}(Y_n)} \tag{1.3}$$

the sequence $(Z_n)_n$ with

$$Z_n = \frac{Y_n - \mathbb{E}[Y_n]}{a_n \sqrt{\mathbb{V}(Y_n)}}$$

satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$. Moreover the sequence

$$\hat{Z}_n = \frac{Y_n - n\varrho_{sc}(I)}{a_n \sqrt{\frac{1}{2\pi^2} \log n}}$$

satisfies the MDP with the same speed, the same rate function, and in the same regime (1.3) *(called the MDP with numerics).*

Before we will prove the MDP for the GUE, we describe the organisation of the next sections. In a first step, we will apply Theorem 1.1 to obtain a MDP for eigenvalues in the bulk of the semicircle law. Next we extend this result to certain families of Hermitian Wigner matrices satisfying tail-condition (T) by means of the Four Moment Theorem due to Tao and Vu. This is the content of Section 2. In Section 3, we show the MDP with numerics for the counting function of Wigner matrices. Moreover we apply recent results of Erdös, Yau and Yin [12] on the localization of eigenvalues and of Dallaporta and Vu [4] in order to prove the MDP without numerics. Section 4 is devoted to discuss the case of symmetric real Wigner matrices as well as the symplectic Gaussian ensemble applying interlacing formulas due to Forrester and Rains, [13]. Finally, in Section 5 we present results for covariance matrices. We prove a universal MDP with numerics for the counting eigenvalue function of covariance matrices.

In [11] we proved a MDP for certain determinantal point processes (DPP), including GUE. Theorem 1.1 follows immediately from an improvement of Theorem 5.2 in [11], which can be easily observed applying the proof of [1, Theorem 4.2.25]. Let Λ be a locally compact Polish space, equipped with a positive Radon measure μ on its Borel σ-algebra. Let $\mathcal{M}_+(\Lambda)$ denote the set of positive σ-finite Radon measures on Λ. A point process is a random, integer-valued $\chi \in \mathcal{M}_+(\Lambda)$, and it is simple if $P(\exists x \in \Lambda : \chi(\{x\}) > 1) = 0$. Let χ be simple. A locally integrable function $\varrho : \Lambda^k \to [0, \infty)$ is called a joint intensity (correlation), if for any mutually disjoint family of subsets $D_1, \dots, D_k$ of Λ

$$\mathbb{E}\big(\prod_{i=1}^k \chi(D_i)\big) = \int_{\prod_{i=1}^k D_i} \varrho_k(x_1, \dots, x_k) d\mu(x_1) \cdots d\mu(x_k),$$

where $\mathbb{E}$ denotes the expectation with respect to the law of the point configurations of χ. A simple point process χ is said to be a *determinantal point process* with kernel K if its joint intensities ϱ_k exist and are given by

$$\varrho_k(x_1, \ldots, x_k) = \det\big(K(x_i, x_j)\big)_{i,j=1,\ldots,k}. \tag{1.4}$$

An integral operator $\mathcal{K} : L^2(\mu) \to L^2(\mu)$ with kernel K given by

$$\mathcal{K}(f)(x) = \int K(x,y) f(y)\, d\mu(y), f \in L^2(\mu),$$

is *admissible* with admissible kernel K if $\mathcal{K}$ is self-adjoint, nonnegative and locally trace-class (for details see [1, 4.2.12]). A standard result is, that an integral compact operator $\mathcal{K}$ with admissible kernel K possesses the decomposition $\mathcal{K}f(x) = \sum_{k=1}^n \eta_k \varphi_k(x) \langle \varphi_k, f \rangle_{L^2(\mu)}$, where the functions φ_k are orthonormal in $L^2(\mu)$, n is either finite or infinite, and $\eta_k > 0$ for all k, leading to

$$K(x,y) = \sum_{k=1}^n \eta_k \varphi_k(x) \varphi_k^*(y), \tag{1.5}$$

an equality in $L^2(\mu \times \mu)$. Moreover, an admissible integral operator $\mathcal{K}$ with kernel K is called *good* with good kernel K if the η_k in (1.5) satisfy $\eta_k \in (0,1]$. If the kernel K of a determinantal point process is (locally) admissible, then it must in fact be good, see [1, 4.2.21].

The following example is the main motivation for discussing determinantal point processes in this paper. Let $(\lambda_1^n, \ldots, \lambda_n^n)$ be the eigenvalues of the GUE (Gaussian unitary ensemble) of dimension n and denote by χ_n the point process $\chi_n(D) = \sum_{i=1}^n 1_{\{\lambda_i^n \in D\}}$. Then χ_n is a determinantal point process with admissible, good kernel $K^{(n)}(x,y) = \sum_{k=0}^{n-1} \Psi_k(x) \Psi_k(y)$, where the functions Ψ_k are the oscillator wave-functions, that is $\Psi_k(x) := \frac{e^{-x^2/4} H_k(x)}{\sqrt{\sqrt{2\pi} k!}}$, where $H_k(x) := (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$ is the k-th Hermite polynomial; see [1, Def. 3.2.1, Ex. 4.2.15].

We will apply the following representation due to [15, Theorem 7]: Suppose χ is a determinantal process with good kernel K of the form (1.5), with $\sum_k \eta_k < \infty$. Let $(I_k)_{k=1}^n$ be independent Bernoulli variables with $P(I_k = 1) = \eta_k$. Set $K_I(x,y) = \sum_{k=1}^n I_k \, \varphi_k(x) \varphi_k^*(y)$, and let χ_I denote the determinantal point process with random kernel K_I. Then χ and χ_I have the same distribution, interpreted as stating that the mixture of determinental processes χ_I has the same distribution as χ. In the following let K be a good kernel and for $D \subset \Lambda$ we write $K_D(x,y) = 1_D(x) K(x,y) 1_D(y)$. Let D be such that K_D is trace-class, with eigenvalues η_k, $k \geq 1$. Then $\chi(D)$ has the same distribution as $\sum_k \xi_k$ where ξ_k are independent Bernoulli random variables with $P(\xi_k = 1) = \eta_k$ and $P(\xi_k = 0) = 1 - \eta_k$.

Theorem 1.4. *Consider a sequence $(\chi_n)_n$ of determinantal point processes on Λ with good kernels K_n. Let D_n be a sequence of measurable subsets of Λ such that $(K_n)_{D_n}$ is trace-class.*

Assume that $(a_n)_n$ *is a sequence of real numbers such that*

$$1 \ll a_n \ll \Big(\sum_{k=1}^n \eta_k^n(1-\eta_k^n)\Big)^{1/2},$$

where η_k^n *are the eigenvalues of* K_n*. Then* $(Z_n)_n$ *with*

$$Z_n := \frac{1}{a_n} \frac{\chi_n(D_n) - \mathbb{E}(\chi_n(D_n))}{\sqrt{\mathbb{V}(\chi_n(D_n))}}$$

satisfies a moderate deviation principle with speed a_n^2 *and rate function* $I(x) = \frac{x^2}{2}$*.*

Remark 1.5. In [16], functional moderate deviations for triangular arrays of certain independent, not identically distributed random variables are considered. Our result, Theorem 1.4, seem to follow from Proposition 1.9 in [16]. Anyhow we prefer to present a direct proof.

Proof of Theorem 1.4. We adapt the proof of [1, Theorem 4.2.25]. We write K_n for the kernel $(K_n)_{D_n}$ and let $S_n := \sqrt{\mathbb{V}(\chi_n(D_n))}$. $\chi_n(D_n)$ has the same distribution as the sum of independent Bernoulli variables ξ_k^n whose parameters η_k^n are the eigenvalues of K_n. We obtain $S_n^2 = \sum_k \eta_k^n(1-\eta_k^n)$ and since K_n is trace-class we can write, for any θ real

$$\begin{aligned}
\log \mathbb{E}\big[e^{\theta a_n^2 Z_n}\big] &= \sum_k \log \mathbb{E}\Big[\exp\Big(\frac{\theta a_n^2(\xi_k^n - \eta_k^n)}{a_n S_n}\Big)\Big] \\
&= -\frac{\theta a_n^2 \sum_k \eta_k^n}{a_n S_n} + \sum_k \log\Big(1 + \eta_k^n\big(e^{a_n^2\theta/(a_n S_n)} - 1\big)\Big).
\end{aligned}$$

For any real θ and n large enough such that $\eta_k^n(e^{\theta a_n/S_n} - 1) \in [0,1]$ we apply Taylor for $\log(1+x)$ and obtain

$$\frac{1}{a_n^2} \log \mathbb{E}\big[e^{\theta a_n^2 Z_n}\big] = \frac{\theta^2 a_n^2 \sum_k \eta_k^n(1-\eta_k^n)}{2 a_n^2 S_n^2} + o\Big(\frac{a_n^3 \sum_k \eta_k^n(1-\eta_k^n)}{a_n^2 S_n^3}\Big).$$

The last term is $o\big(\frac{a_n}{S_n}\big)$. Applying the Theorem of Gärtner-Ellis, [7, Theorem 2.3.6], the result follows. $\square$

Proof of Theorem 1.1. Now the first statement of Theorem 1.1 follows since $\mathbb{V}(\chi_n(D_n)) \to \infty$, see [1, Cor. 4.2.27]. In particular for any $I = [y, \infty)$ with $y \in (-2, 2)$

$$\tilde{Z}_n := \frac{N_I(W_n') - \mathbb{E}[N_I(W_n')]}{a_n \sqrt{\mathbb{V}(N_I(W_n'))}}$$

satisfies the MDP. The MDP with numerics (see Remark 1.2) follows, since the sequences $(\tilde{Z}_n)_n$ and $(\hat{Z}_n)_n$ are exponentially equivalent in the sense of definition [7, Definition 4.2.10], and hence the result follows from [7, Theorem 4.2.13]: Let

$$Z_n'' := \frac{N_I(W_n') - n\varrho_{sc}(I)}{a_n \sqrt{\mathbb{V}(N_I(W_n'))}}.$$

Since $|\tilde{Z}_n - Z''_n| = \left|\frac{\mathbb{E}[N_I(W'_n)] - n\varrho_{sc}(I)}{a_n\sqrt{\mathbb{V}(N_I(W'_n))}}\right| \to 0$ as $n \to \infty$, $(\tilde{Z}_n)_n$ and $(Z''_n)_n$ are exponentially equivalent. Moreover by Taylor we obtain $|Z''_n - \hat{Z}_n| = \frac{o(1)}{a_n} \frac{N_I(W'_n) - n\varrho_{sc}(I)}{\sqrt{\mathbb{V}(N_I(W'_n))}}$ and the MDP for $(Z''_n)_n$ implies $\limsup_{n\to\infty} \frac{1}{a_n^2} \log P\big(|Z''_n - \hat{Z}_n| > \varepsilon\big) = -\infty$ for any $\varepsilon > 0$. $\square$

2. Moderate deviations for eigenvalues in the bulk

Under certain conditions on i it was proved in [14] that the i-th eigenvalue λ_i of the GUE W'_n satisfies a CLT. Consider $t(x) \in [-2, 2]$ defined for $x \in [0, 1]$ by

$$x = \int_{-2}^{t(x)} d\varrho_{sc}(t) = \frac{1}{2\pi} \int_{-2}^{t(x)} \sqrt{4 - x^2}\, dx.$$

Then for $i = i(n)$ such that $i/n \to a \in (0, 1)$ as $n \to \infty$ (i.e. λ_i is eigenvalue in the bulk), $\lambda_i(W'_n)$ satisfies a CLT:

$$\sqrt{\frac{4 - t(i/n)^2}{2}} \frac{\lambda_i(W'_n) - t(i/n)}{\frac{\sqrt{\log n}}{n}} \to N(0, 1) \tag{2.1}$$

for $n \to \infty$. Remark that $t(i/n)$ is sometimes called the *classical or expected location* of the i-th eigenvalue. The standard deviation is $\frac{\sqrt{\log n}}{\pi\sqrt{2}} \frac{1}{n\varrho_{sc}(t(i/n))}$. Note that from the semicircular law, the factor $\frac{1}{n\varrho_{sc}(t(i/n))}$ is the mean eigenvalue spacing.

The proof in [14] is achieved by the tight relation between eigenvalues and the counting function expressed by the elementary equivalence, for $I(y) = [y, \infty)$, $y \in \mathbb{R}$,

$$N_{I(y)}(W_n) \le n - i \text{ if and only if } \lambda_i \le y. \tag{2.2}$$

Hence the theorem due to Costin and Lebowitz as well as Soshnikov, see [18], can be applied. Moreover the proof in [14] relies on fine asymptotics for the Airy function and the Hermite polynomials due to [5].

Our first result in this Section is a corresponding MDP for λ_i in the bulk:

Theorem 2.1. *Consider the GUE matrix $W'_n = \frac{1}{\sqrt{n}} M'_n$. Consider $i = i(n)$ such that $i/n \to a \in (0, 1)$ as $n \to \infty$. If λ_i denotes the eigenvalue number i in the GUE matrix W'_n it holds that for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll \sqrt{\log n}$ the sequence $(X'_n)_n$ with*

$$X'_n = \sqrt{\frac{4 - t(i/n)^2}{2}} \frac{\lambda_i(W'_n) - t(i/n)}{a_n \frac{\sqrt{\log n}}{n}}$$

satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$.

Interesting enough, this result can be extended to large families of Hermitian Wigner matrices satisfying tail-condition (T) by means of the Four Moment Theorem of Tao and Vu:

Theorem 2.2. *Consider a Hermitian Wigner matrix* $W_n = \frac{1}{\sqrt{n}} M_n$ *whose entries satisfy tail-condition* (T) *and match the corresponding entries of GUE up to order 4. Consider* $i = i(n)$ *such that* $i/n \to a \in (0,1)$ *as* $n \to \infty$*. If* λ_i *denotes the eigenvalue number* i *of* W_n *it holds that for any sequence* $(a_n)_n$ *of real numbers such that* $1 \ll a_n \ll \sqrt{\log n}$*, the sequence* $(X_n)_n$ *with*

$$X_n = \sqrt{\frac{4 - t(i/n)^2}{2}} \frac{\lambda_i(W_n) - t(i/n)}{a_n \frac{\sqrt{\log n}}{n}}$$

satisfies a MDP with speed a_n^2 *and rate function* $I(x) = \frac{x^2}{2}$.

Remark 2.3. In [14] a CLT at the edge of the spectrum was considered also. The proof applies the result of Costin and Lebowitz as well as fine asymptotics presented in [5]. Consider $i \to \infty$ such that $i/n \to 0$ as $n \to \infty$ and consider λ_{n-i}, eigenvalue number $n-i$ in the GUE or Hermitian Wigner case. A CLT for the rescaled λ_{n-i} is stated in [14, Theorem 1.2]. We would be able to formulate and prove a MDP for eigenvalue λ_{n-i}, but it is not the main focus of this paper. We omit this.

Proof of Theorem 2.1. The proof is oriented to the proof of [14, Theorem 1.1] and will apply the precise asymptotic behaviour of the expectation and of the variance of the counting function $N_I(W_n')$, see (1.1), which is a reformulation of [14, Lemma 2.1-2.3]. Let P_n denote the probability of the GUE determinantal point processes, and set

$$I_n := \left[t(i/n) + \xi\, a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}, \infty \right).$$

Now we apply relation (2.2) and obtain

$$\begin{aligned} P_n\Bigg(\frac{\lambda_i(W_n') - t(i/n)}{a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4-t(i/n)^2}}} \leq \xi \Bigg) &= P_n\Bigg(\lambda_i(W_n') \leq \xi\, a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}} + t(i/n) \Bigg) \\ = \quad P_n\big(N_{I_n}(W_n') \leq n - i \big) &= P_n\Bigg(\frac{N_{I_n}(W_n') - \mathbb{E}[N_{I_n}(W_n')]}{a_n\, (\mathbb{V}(N_{I_n}(W_n')))^{1/2}} \leq \frac{n - i - \mathbb{E}[N_{I_n}(W_n')]}{a_n\, (\mathbb{V}(N_{I_n}(W_n')))^{1/2}} \Bigg). \end{aligned}$$

Since $i/n \to a \in (0,1)$, by definition of $t(x)$ we obtain $t(i/n) \in (-2,2)$. Moreover since $a_n \ll \sqrt{\log n}$ we have $\xi\, a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4-t(i/n)^2}} + t(i/n) \in (-2,2)$ for n large. Therefore with (1.1) we have for n sufficiently large that

$$\mathbb{E}[N_{I_n}(W_n')] = n\, \varrho_{sc}(I_n) + O\big(\frac{\log n}{n}\big) \text{ and } \mathbb{V}(N_{I_n}(W_n')) = \big(\frac{1}{2\pi^2} + o(1)\big) \log n.$$

With $b(n) := a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4-t(i/n)^2}}$ and $f_n(t(i/n)) := t(i/n) + \xi\, b(n)$ we get from symmetry

$$\varrho_{sc}(I_n) = \int_{f_n(t(i/n))}^{\infty} \varrho_{sc}(x)\, dx = \frac{1}{2} - \int_0^{f_n(t(i/n))} \varrho_{sc}(x)\, dx = 1 - \int_{-2}^{f_n(t(i/n))} \varrho_{sc}(x)\, dx.$$

Now

$$\int_{-2}^{f_n(t(i/n))} \varrho_{sc}(x)\,dx = \int_{-2}^{t(i/n)} \varrho_{sc}(x)\,dx + \int_{t(i/n)}^{f_n(t(i/n))} \varrho_{sc}(x)\,dx = \frac{i}{n} + \int_{t(i/n)}^{f_n(t(i/n))} \varrho_{sc}(x)\,dx$$

and

$$\int_{t(i/n)}^{f_n(t(i/n))} \varrho_{sc}(x)\,dx = \xi b(n) \frac{1}{2\pi}\sqrt{4 - t(i/n)^2} + O\big(b(n)^2\big).$$

Summarizing we obtain

$$n\,\varrho_{sc}(I_n) = n - i - \xi\, a_n \sqrt{\log n}\frac{1}{\sqrt{2}\pi} + O\big(\frac{a_n^2 \log n}{n}\big)$$

and therefore

$$\frac{n - i - \mathbb{E}[N_{I_n}(W_n')]}{a_n\,(\mathbb{V}(N_{I_n}(W_n')))^{1/2}} = \xi + \varepsilon(n),$$

where $\varepsilon(n) \to 0$ as $n \to \infty$. By Theorem 1.1 we obtain for every $\xi < 0$

$$\lim_{n\to\infty} \frac{1}{a_n^2} \log P_n\left(\frac{\lambda_i(W_n') - t(i/n)}{a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}} \le \xi \right) = -\frac{\xi^2}{2}. \tag{2.3}$$

With

$$P_n\left(\frac{\lambda_i(W_n') - t(i/n)}{a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}} \ge \xi \right) = P_n\big(N_{I_n}(W_n') \ge n - i + 1\big)$$

the same calculations lead, for every $\xi > 0$, to

$$\lim_{n\to\infty} \frac{1}{a_n^2} \log P_n\left(\frac{\lambda_i(W_n') - t(i/n)}{a_n \frac{\sqrt{\log n}}{n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}} \ge \xi \right) = -\frac{\xi^2}{2}. \tag{2.4}$$

Hence the conclusion follows: see for example [7, Proof of Theorem 2.2.3]. To be more precise we apply the preceding results (2.3) and (2.4) with Theorem 4.1.11 in [7], which allows us to derive a MDP from the limiting behaviour of probabilities for a basis of topology. For the latter, we choose all open intervals (a, b), where at least one of the endpoints is finite and where none of the endpoints is zero. Denote the family of such intervals by $\mathcal{U}$. From (2.3) and (2.4), it follows for each $U = (a, b) \in \mathcal{U}$,

$$\mathcal{L}_U := \lim_{n\to\infty} \frac{1}{a_n^2} \log P\big(X_n' \in U\big) = \begin{cases} b^2/2 & : \quad a < b < 0 \\ 0 & : \quad a < 0 < b \\ a^2/2 & : \quad 0 < a < b \end{cases}$$

By [7, Theorem 4.1.11], $(X_n')_n$ satisfies a weak MDP with speed a_n^2 and rate function

$$t \mapsto \sup_{U \in \mathcal{U}; t \in U} \mathcal{L}_U = \frac{t^2}{2}.$$

With (2.4), it follows that $(X_n')_n$ is exponentially tight, hence by Lemma 1.2.18 in [7], $(X_n')_n$ satisfies the MDP with the same speed and the same good rate function. This completes the proof. $\square$

To extend the result of Theorem 2.1 to Hermitian Wigner matrices satisfying tail-condition (T), we will apply the Four Moment Theorem (for the bulk of the spectra of Wigner matrices), see [21, Theorem 15].

Theorem 2.4 (Four Moment Theorem due to Tao and Vu)**.** *There is a small positive constant c_0 such that for every $0 < \varepsilon < 1$ and $k \geq 1$ the following holds. Let M_n and M_n' be two Hermitian Wigner matrices satisfying tail-condition (T). Assume furthermore that for any $1 \leq i < j \leq n$, Z_{ij} and Z_{ij}' match to order 4 and for any $1 \leq i \leq n$, Y_i and Y_i' match to order 2. Set $A_n := \sqrt{n} M_n$ and $A_n' := \sqrt{n} M_n'$, and let $G : \mathbb{R}^k \to \mathbb{R}$ be a smooth function obeying the derivative bounds $|\nabla^j G(x)| \leq n^{c_0}$ for all $0 \leq j \leq 5$ and $x \in \mathbb{R}^k$. Then for any $\varepsilon n \leq i_1 < i_2 \cdots < i_k \leq (1-\varepsilon)n$, and for n sufficiently large depending on ε, k and the constants C, C' in tail-condition (T), we have*

$$|\mathbb{E}\big(G(\lambda_{i_1}(A_n), \dots, \lambda_{i_k}(A_n))\big) - \mathbb{E}\big(G(\lambda_{i_1}(A_n'), \dots, \lambda_{i_k}(A_n'))\big)| \leq n^{-c_0}. \tag{2.5}$$

Applying this Theorem for the special case when M_n' is GUE, one obtains [21, Corollary 18]:

Corollary 2.5. *Let M_n be a Hermitian Wigner matrix whose atom distribution ξ satisfies $\mathbb{E}\xi^3 = 0$ and $\mathbb{E}\xi^4 = \frac{3}{4}$ and tail-condition (T), and M_n' be a random matrix sampled from GUE. Then with G, A_n, A_n' as in the previous theorem, and n sufficiently large, one has*

$$|\mathbb{E}\big(G(\lambda_{i_1}(A_n), \dots, \lambda_{i_k}(A_n))\big) - \mathbb{E}\big(G(\lambda_{i_1}(A_n'), \dots, \lambda_{i_k}(A_n'))\big)| \leq n^{-c_0}. \tag{2.6}$$

Now the universality of the MDP in Theorem 2.2 follows along the lines of the proof of [21, Corollary 21].

Proof of Theorem 2.2. Let M_n be a Hermitian Wigner matrix whose entries satisfy tail-condition (T) and match the corresponding entries of GUE up to order 4. Let i, a and $t(\cdot)$ be as in the statement of the Theorem, and let c_0 be as in Corollary 2.5. Then [21, (18)] says that

$$P_n\big(\lambda_i(A_n') \in I_-\big) - n^{-c_0} \leq P_n\big(\lambda_i(A_n) \in I\big) \leq P_n\big(\lambda_i(A_n') \in I_+\big) + n^{-c_0} \tag{2.7}$$

for all intervals $[b, c]$, and n sufficiently large depending on i and the constants C, C' of tail-condition (T). Here $I_+ := [b - n^{-c_0/10}, c + n^{-c_0/10}]$ and $I_- := [b + n^{-c_0/10}, c - n^{-c_0/10}]$. We present the argument of proof of (2.7) just to make the presentation more self-contained. One can find a smooth bump function $G : \mathbb{R} \to \mathbb{R}_+$ which is equal to one on the smaller interval I_- and vanishes outside the larger interval I_+. It follows that $P_n\big(\lambda_i(A_n) \in I\big) \leq \mathbb{E}G(\lambda_i(A_n))$ and $\mathbb{E}G(\lambda_i(A_n')) \leq P_n\big(\lambda_i(A_n') \in I\big)$. One can choose G to obey the condition $|\nabla^j G(x)| \leq n^{c_0}$ for $j = 0, \dots, 5$ and hence by Corollary 2.5 one gets

$$|\mathbb{E}G(\lambda_i(A_n)) - \mathbb{E}G(\lambda_i(A_n'))| \leq n^{-c_0}.$$

Therefore the second inequality in (2.7) follows from the triangle inequality. The first inequality is proven similarly.

Now for n sufficiently large we consider the interval $I_n := [b_n, c_n]$ with

$$b_n := b\, a_n \sqrt{\log n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}} + nt(i/n) \text{ and } c_n := c\, a_n \sqrt{\log n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}} + nt(i/n)$$

with $b, c \in \mathbb{R}$, $b \leq c$. Then for X_n defined as in the statement of the Theorem we have

$$P_n\big(X_n \in [b, c]\big) = P_n\left(\frac{\lambda_i(A_n) - nt(i/n)}{a_n \sqrt{\log n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}} \in [b, c]\right) = P_n\big(\lambda_i(A_n) \in I_n\big).$$

With (2.7) and [7, Lemma 1.2.15] we obtain

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \log P_n\big(X_n \in [b, c]\big) \leq \max\left(\limsup_{n\to\infty} \frac{1}{a_n^2} \log P_n\big(\lambda_i(A_n') \in (I_n)_+\big); \limsup_{n\to\infty} \frac{1}{a_n^2} \log n^{-c_0}\right).$$

For the first object we have

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \log P_n\big(\lambda_i(A_n') \in (I_n)_+\big) = \limsup_{n\to\infty} \frac{1}{a_n^2} \log P_n\left(\frac{\lambda_i(A_n') - nt(i/n)}{a_n \sqrt{\log n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}} \in [b - \eta(n), c + \eta(n)]\right)$$

with $\eta(n) = n^{-c_0/10}\big(a_n \sqrt{\log n} \frac{\sqrt{2}}{\sqrt{4 - t(i/n)^2}}\big)^{-1} \to 0$ as $n \to \infty$. Since $c_0 > 0$ and $\log n / a_n^2 \to \infty$ for $n \to \infty$ by assumption, applying Theorem 2.1 we have

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \log P_n\big(X_n \in [b, c]\big) \leq - \inf_{x \in [b,c]} \frac{x^2}{2}.$$

Applying the first inequality in (2.7) in the same manner we also obtain the lower bound

$$\liminf_{n\to\infty} \frac{1}{a_n^2} \log P_n\big(X_n \in [b, c]\big) \geq - \inf_{x \in [b,c]} \frac{x^2}{2}.$$

Finally the argument in the last part of the proof of Theorem 2.1 can be repeated to obtain the MDP for $(X_n)_n$. $\square$

3. Universality of the Moderate deviations

In this section we proof Theorem 1.3. As announced, we will show that the MDP behaviour of eigenvalues in the bulk of the GUE (Theorem 2.1) extended to Hermitian Wigner matrices (Theorem 2.2) leads to the MDP with numerics for the counting function of eigenvalues of Hermitian Wigner matrices.

Proof of Theorem 1.3. For every $\xi \in \mathbb{R}$ we obtain that

$$P_n\big(\hat{Z}_n \leq \xi\big) = P_n\big(N_{I(y)}(W_n) \leq n - i_n\big)$$

with $i_n := n\varrho_{sc}((-\infty, y]) - \xi\, a_n \sqrt{\frac{1}{2\pi^2} \log n}$. Hence using (2.2) it follows

$$P_n\big(\hat{Z}_n \leq \xi\big) = P_n\big(\lambda_{i_n} \leq y\big) = P_n\left(\sqrt{\frac{4 - t(i_n/n)^2}{2}} \frac{\lambda_{i_n}(W_n) - t(i_n/n)}{a_n \frac{\sqrt{\log n}}{n}} \leq \xi_n\right)$$

with $\xi_n := \sqrt{\frac{4-t(i_n/n)^2}{2}} \frac{y-t(i_n/n)}{a_n \frac{\sqrt{\log n}}{n}}$. Now

$$\frac{i_n}{n} = \varrho_{sc}((-\infty, y]) - \frac{\xi a_n \sqrt{\frac{1}{2\pi^2}\log n}}{n} \to \varrho_{sc}((-\infty, y]) \in (0,1)$$

for $n \to \infty$. We will prove that $\xi_n = \xi + o(1)$. Applying Theorem 2.2, it follows that

$$\lim_{n\to\infty} \frac{1}{a_n^2} \log P_n\big(\hat{Z}_n \le \xi\big) = -\frac{\xi^2}{2}$$

for all $\xi < 0$. With $P_n\big(\hat{Z}_n \ge \xi\big) = P_n\big(N_{I(y)}(W_n) \ge n - i_n\big) = P_n\big(\lambda_{i_n+1} \ge y\big)$ the same calculations will lead, for any $\xi > 0$, to

$$\lim_{n\to\infty} \frac{1}{a_n^2} \log P_n\big(\hat{Z}_n \ge \xi\big) = -\frac{\xi^2}{2}.$$

The MDP for $(\hat{Z}_n)_n$ (the MDP with numerics) now follows along the lines of the proof of Theorem 2.1 (topological argument).

The MDP for $(Z_n)_n$ follows by the arguments given in the proof of Theorem 1.1, using the deep fact that the expectation $\mathbb{E}[Y_n]$ and the variance $\mathbb{V}(Y_n)$ of the eigenvalue counting function have identical behaviours to the ones for GUE matrices:

$$\mathbb{E}[Y_n] = n\varrho_{sc}(I(y)) + o(1) \text{ and } \mathbb{V}(Y_n) = \big(\frac{1}{2\pi^2} + o(1)\big)\log n.$$

This result is established in [4, Theorem 2], applying strong localization of the eigenvalues of Wigner matrices, a recent result from [12].

Finally we prove that $\lim_{n\to\infty} \xi_n = \xi$. We obtain

$$\begin{aligned} t(i_n/n) &= t\Bigg(\varrho_{sc}((-\infty, y]) - \frac{1}{n}\,\xi\, a_n \sqrt{\frac{1}{2\pi^2}\log n}\Bigg) \\ &= t\big(\varrho_{sc}((-\infty, y])\big) - \frac{1}{n}\,\xi\, a_n \sqrt{\frac{1}{2\pi^2}\log n}\, t'\big(\varrho_{sc}((-\infty, y])\big) + o\big(\frac{a_n\sqrt{\log n}}{n}\big) \\ &= y - \xi\frac{1}{n} a_n \sqrt{\log n}\frac{\sqrt{2}}{\sqrt{4-y^2}} + o\big(\frac{a_n\sqrt{\log n}}{n}\big). \end{aligned}$$

Hence $\frac{y-t(i_n/n)}{a_n\frac{\sqrt{\log n}}{n}} = \xi\frac{\sqrt{2}}{\sqrt{4-y^2}} + o(1)$ and with $\lim_{n\to\infty}\sqrt{\frac{4-t(i_n/n)^2}{2}} = \sqrt{\frac{4-y^2}{2}}$ it follows that $\lim_{n\to\infty}\xi_n = \xi$. □

4. Symmetric Wigner matrices and the GSE

In this section, we indicate how the preceding results for Hermitian Wigner matrices can be stated and proved for real Wigner symmetric matrices. Moreover we consider a Gaussian symplectic ensemble (GSE). Real Wigner matrices are random symmetric matrices M_n of size n such that, for $i < j$, $(M_n)_{ij}$ are i.i.d. with mean zero and variance one, $(M_n)_{ii}$ are i.i.d. with

mean zero and variance 2. As already mentioned, the case where the entries are Gaussian is the GOE. As in Section 1 and 2, the main issue is to establish our conclusions for the GOE. On the level of CLT, this was developed in [17] by means of the famous *interlacing formulas* due to Forrester and Rains, [13], that relates the eigenvalues of different matrix ensembles. The following relation holds between matrix ensembles:

$$\mathrm{GUE}_n = \mathrm{even}\big(\mathrm{GOE_n} \cup \mathrm{GOE_{n+1}}\big). \tag{4.1}$$

The statement is: Take two independent (!) matrices from the GOE: one of size $n \times n$ and one of size $(n+1) \times (n+1)$. Superimpose the $2n+1$ eigenvalues on the real line and then take the n even ones. They have the same distribution as the eigenvalues of a $n \times n$ matrix from the GUE. If $M_n^{\mathbb{R}}$ denotes a GOE matrix and $W_n^{\mathbb{R}} := \frac{1}{\sqrt{n}} M_n^{\mathbb{R}}$, first we will prove a MDP for

$$Z_n^{\mathbb{R}} := \frac{N_{I_n}(W_n^{\mathbb{R}}) - \mathbb{E}[N_{I_n}(W_n^{\mathbb{R}})]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))}} \tag{4.2}$$

for any $1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))}$, I_n an interval in $\mathbb{R}$, with speed a_n^2 and rate $x^2/2$. Let within this section $M_n^{\mathbb{C}}$ denote a GUE matrix and $W_n^{\mathbb{C}}$ the corresponding normalized matrix. The nice consequences of (4.1) were already suitably developed in [17]: applying Cauchy's interlacing theorem one can write

$$N_{I_n}(W_n^{\mathbb{C}}) = \frac{1}{2}\big[N_{I_n}(W_n^{\mathbb{R}}) + N'_{I_n}(W_n^{\mathbb{R}}) + \eta'_n(I_n)\big], \tag{4.3}$$

where one obtains GOE'_n in $N'_{I_n}(W_n^{\mathbb{R}})$ from GOE_{n+1} by considering the principle sub-matrix of GOE_{n+1} and $\eta'_n(I_n)$ takes values in $\{-2,-1,0,1,2\}$. Note that $N_{I_n}(W_n^{\mathbb{R}})$ and $N'_{I_n}(W_n^{\mathbb{R}})$ are independent because GOE_{n+1} and GOE_n denote independent matrices from the GOE. We obtain

$$\begin{aligned} Z_n^{\mathbb{C}} := \frac{N_{I_n}(W_n^{\mathbb{C}}) - \mathbb{E}[N_{I_n}(W_n^{\mathbb{C}})]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}}))}} &= \frac{N_{I_n}(W_n^{\mathbb{R}}) - \mathbb{E}[N_{I_n}(W_n^{\mathbb{R}})]}{a_n 2 \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}}))}} + \frac{N'_{I_n}(W_n^{\mathbb{R}}) - \mathbb{E}[N'_{I_n}(W_n^{\mathbb{R}})]}{a_n 2 \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}}))}} \\ &+ \frac{\eta'_n(I_n) - \mathbb{E}[\eta'_n(I_n)]}{a_n 2 \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}}))}} =: X_n + Y_n + \varepsilon_n \end{aligned} \tag{4.4}$$

Now we can make use of the MDP for $(Z_n^{\mathbb{C}})_n$, Theorem 1.1. Using the independence of X_n and Y_n in (4.4), as well as the fact that the third summand can be estimated by $|\varepsilon_n| \le \frac{2}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}}))}}$, we obtain for every θ

$$-\frac{|2\theta|}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}}))}} + 2\frac{1}{a_n^2} \log \mathbb{E} e^{\theta a_n^2 X_n} \le \frac{1}{a_n^2} \log \mathbb{E} e^{\theta a_n^2 Z_n} \le \frac{|2\theta|}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}}))}} + 2\frac{1}{a_n^2} \log \mathbb{E} e^{\theta a_n^2 X_n}.$$

Applying the Theorem of Gärtner-Ellis, [7, Theorem 2.3.6], the MDP for $(X_n)_n$ with speed a_n^2 and rate x^2 follows for all $(I_n)_n$ with $\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) \to \infty$. Hence we have proved the MDP-version of [17, Lemma 2], that $\big(\frac{N_{I_n}(W_n^{\mathbb{R}}) - \mathbb{E}[N_{I_n}(W_n^{\mathbb{R}})]}{a_n \sqrt{2} \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))}}\big)_n$ satisfies an MDP with rate $x^2/2$ if $\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) \to \infty$. The interlacing formula (4.3) leads to $2\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) + O(1) = \mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))$ if $\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) \to \infty$. Therefore $(Z_n^{\mathbb{R}})_n$ satisfies the MDP with speed a_n^2 and rate function $x^2/2$.

The proof of Theorem 2.1 can simply be adapted to the GOE. Since the Four Moment Theorem also applies for real symmetric matrices, with an analog of [4, Lemma 5] in hand we obtain:

Theorem 4.1. *Consider a real symmetric Wigner matrix $W_n = \frac{1}{\sqrt{n}} M_n$ whose entries satisfy tail-condition (T) and match the corresponding entries of GOE up to order 4.*

(1) *Consider $i = i(n)$ such that $i/n \to a \in (0,1)$ as $n \to \infty$. Denote the eigenvalue number i of W_n by λ_i. Let $(a_n)_n$ be a sequence of real numbers such that $1 \ll a_n \ll \sqrt{\log n}$. Then the sequence $(X_n)_n$ with $X_n := \frac{\lambda_i - t(i/n)}{a_n \frac{\sqrt{\log n}}{n} \frac{2}{\sqrt{4-t(i/n)^2}}}$ universally satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$.* **(2)** *For any $y \in (-2,2)$ and $I(y) = [y, \infty)$, the rescaled eigenvalue counting function $N_{I(y)}(W_n)$ universally satisfies the MDP as in Theorem 1.3.*

Finally we consider the Gaussian Symplectic Ensemble (GSE). Here the following relation holds between matrix ensembles: $\mathrm{GSE}_n = \mathrm{even}\big(\mathrm{GOE}_{2n+1}\big)\frac{1}{\sqrt{2}}$. The multiplication by $\frac{1}{\sqrt{2}}$ denotes scaling the $(2n+1) \times (2n+1)$ GOE matrix by the factor $\frac{1}{\sqrt{2}}$. Let $x_1 < x_2 < \cdots < x_n$ denote the ordered eigenvalues of an $n \times n$ matrix from the GSE and let $y_1 < y_2 < \cdots < y_{2n+1}$ denote the ordered eigenvalues of an $(2n+1) \times (2n+1)$ matrix from the GOE. Then it follows that $x_i = y_{2i}/\sqrt{2}$ in distribution. Hence the MDP for the i-th eigenvalue of the GSE follows directly from the GOE case.

Theorem 4.2. *Consider the GSE matrix $W_n^{\mathbb{H}} = \frac{1}{\sqrt{n}} M_n^{\mathbb{H}}$. Consider $i = i(n)$ such that $i/n \to a \in (0,1)$ as $n \to \infty$. If λ_i denotes the eigenvalue number i in the GSE matrix, it holds that for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll \sqrt{\log n}$ the sequence*

$$\sqrt{4 - t(i/n)^2}\,\frac{\lambda_i(W_n^{\mathbb{H}}) - t(i/n)}{a_n \frac{\sqrt{\log n}}{n}}$$

satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$. For any $y \in (-2,2)$ and $I(y) = [y, \infty)$, the rescaled eigenvalue counting function $N_{I(y)}(W_n^{\mathbb{H}})$ satisfies the MDP as in Theorem 1.1.

5. Moderate deviations for covariance matrices

In this section we briefly present the analogous results for covariance matrices. They rely on [19], where Gaussian fluctuations of individual eigenvalues in complex sample covariance matrices were considered, as well as on the Four Moment Theorem for random covariance matrices due to Tao and Vu ([22]). Let $p = p(n)$ and n be integers such that $p \geq n$ and $\lim_{n\to\infty} \frac{p}{n} = \gamma \in [1, \infty)$. Let X be a random $p \times n$ matrix with complex entries X_{ij} such that they are identically distributed and independent, have mean zero and variance 1. Assume moreover that the following condition is fulfilled:

Moment-condition (M): X satisfies moment-condition (M), if there exists $C_0 \geq 2$ and $C > 0$ such that $\sup_{ij} \mathbb{E}[|X_{ij}|^{C_0}] \leq C$.

Then $W := W_{p,n} := \frac{1}{n} X^* X$ is called *covariance matrix*. Hence it has at most p non zero eigenvalues, which are real and nonnegative, denoted by $0 \leq \lambda_1(W) \leq \cdots \leq \lambda_p(W)$. We abbreviate $\lambda_i(W)$ as λ_i. If the entries are Gaussian random variables, $W_{p,n}$ is called Laguerre Unitary Ensemble (LUE) or complex Gaussian Wishart ensemble. LUE matrices will be denoted by $W' = W'_{p,n}$. In this case, the eigenvalues form a determinantal point process with admissible, good kernel given in terms of Laguerre polynomials; see for example [19]. With Theorem 1.4 we obtain a MDP for the counting function $N_{I_n}(W'_{p,n})$ for intervals I_n with $\mathbb{V}(N_{I_n}(W'_{p,n})) \to \infty$ for $n \to \infty$. The classical Marchenko-Pastur theorem states that as $n \to \infty$ such that $\frac{p}{n} \to \gamma \geq 1$, almost surely $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} \to \mu_\gamma$ in distribution, where μ_γ is the Marchenko-Pastur law with density $d\mu_\gamma(x) = \frac{1}{2\pi x} \sqrt{(x-\alpha)(\beta - x)} 1_{[\alpha,\beta]}(x)\, dx$, where $\alpha = (\sqrt{\gamma} - 1)^2$ and $\beta = (\sqrt{\gamma}+1)^2$.

Let

$$\alpha_{p,n} := \left(\sqrt{\frac{p}{n}} - 1 \right)^2, \ \beta_{p,n} := \left(\sqrt{\frac{p}{n}} + 1 \right)^2$$

and $I_n := [t_n, \infty)$ with $t_n \leq \beta_{n,p} - \delta$ for some $\delta > 0$. Then with [19, Lemma 3], the variance of the number of eigenvalues of $W'_{p,n}$ in I_n satisfies

$$\mathbb{V}(N_{I_n}(W'_{p,n})) = \frac{1}{2\pi^2} \log n (1 + o(1)). \tag{5.1}$$

Moreover with [19, Lemma 1], the expected number of eigenvalues of $W'_{p,n}$ in $I_n = [t_n, \infty)$ with $t_n \to t \in (\alpha, \beta)$ satisfies

$$\mathbb{E}[N_{I_n}(W'_{p,n})] = n \int_{t_n}^{\beta_{p,n}} \mu_{p,n}(x)\, dx\, (1 + o(1)), \tag{5.2}$$

where $\mu_{p,n}(x) := \frac{1}{2\pi x} \sqrt{(x - \alpha_{p,n})(\beta_{p,n} - x)} 1_{[\alpha_{p,n}, \beta_{p,n}]}(x)$. The proof of Theorem 1.4 leads to an MDP (with numerics) for

$$\left(\frac{N_{I_n}(W'_{p,n}) - n \int_{t_n}^{\beta_{p,n}} \mu_{p,n}(x)\, dx}{a_n \sqrt{\frac{1}{2\pi^2} \log n}} \right)_n$$

for every $I_n = [t_n, \infty)$ with $t_n \to t \in (\alpha, \beta)$ and all $1 \ll a_n \ll \sqrt{\log n}$. Let

$$G(t) := \int_{\alpha_{p,n}}^t \mu_{p,n}(x)\, dx \text{ for } \alpha_{p,n} \leq t \leq \beta_{p,n}.$$

Arguing as in Section 2, the following MDP can be achieved:

Theorem 5.1. *Consider the LUE matrix $W'_{p,n}$. Let $t := t_{n,i} = G^{-1}(i/n)$ with $i = i(n)$ such that $\frac{i}{n} \to a \in (0,1)$ as $n \to \infty$. Then as $\frac{p}{n} \to \gamma \geq 1$ the sequence $(X_n)_n$ with*

$$X_n := \frac{\sqrt{2}\pi \mu_{p,n}(t)(\lambda_i - t)}{a_n \frac{\sqrt{\log n}}{n}} \tag{5.3}$$

satisfies an MDP for any $1 \ll a_n \ll \sqrt{\log n}$ with rate $x^2/2$.

Proof. Along the lines of the proof of Theorem 2.1, the main step is to calculate $\mathbb{E}\big[N_{I_n}(W'_{p,n})\big]$ for $I_n=[t+\xi b_n,\infty)$ with $b_n=a_n\frac{\sqrt{\log n}}{n}\frac{1}{\pi\sqrt{2}\mu_{p,n}(t)}$. Using the Taylor expansion for $\int_{t+\xi b_n}^{\cdot}\mu_{p,n}(x)\,dx$ we obtain

$$\mathbb{E}\big[N_{I_n}(W'_{p,n})\big]=n\int_{t+\xi b_n}^{\beta_{p,n}}\mu_{p,n}(x)\,dx\,(1+o(1))=n-i-\xi a_n\sqrt{\log n}\frac{1}{\sqrt{2}\pi}+o(1).$$

The statement follows step by step along the proof of Theorem 2.1. $\square$

As was done for Wigner matrices, one can extend the last Theorem to general covariance matrices $W_{p,n}$ whose entries satisfy the moment-condition (M) and match the corresponding entries of LUE up to order 4. Namely, Tao and Vu extended their Four Moment Theorem to the case of covariance matrices in [22, Theorem 6]. We apply it in the same way as for Wigner matrices. Finally we end up with the following universality result:

Theorem 5.2. *Let $W_{p,n}$ be a covariance matrix whose entries satisfy moment-condition (M) and match the corresponding entries of LUE up to order 4. Then, for any $I_n=[t_n,\infty)$ with $t_n\to t\in(\alpha,\beta)$ and all $1\ll a_n\ll\sqrt{\log n}$ the sequence $(\hat{Z}_n)_n$ with*

$$\hat{Z}_n=\frac{N_{I_n}(W_{p,n})-n\int_{t_n}^{\beta_{p,n}}\mu_{p,n}(x)\,dx}{a_n\sqrt{\frac{1}{2\pi^2}\log n}}$$

satisfies a MDP (with numerics) with speed a_n^2 and rate function $I(x)=\frac{x^2}{2}$.

Remark 5.3. Since a version of the Erdös-Yau-Yin rigidity theorem for covariance matrices is not yet proved, the MDP "without numerics" for the eigenvalue counting function for non-Gaussian covariance matrices is not stated.

Proof. For every $\xi\in\mathbb{R}$ we obtain with X_n defined in (5.3) (where $\lambda_i=\lambda_i(W_{p,n})$) that

$$P_n\big(\hat{Z}_n\le\xi\big)=P_n\big(X_n\le\xi_n\big)$$

with

$$\xi_n=\sqrt{2}\pi\mu_{p,n}(G^{-1}(i_n/n))\frac{t_n-G^{-1}(i_n/n)}{a_n\frac{\sqrt{\log n}}{n}}$$

and $i_n:=n\int_{\alpha_{p,n}}^{t_n}\mu_{p,n}(x)\,dx-\xi a_n\sqrt{\frac{1}{2\pi^2}\log n}$. Now $i_n/n\to\mu_\gamma(-\infty,t])\in(0,1)$, since $t\in(\alpha,\beta)$. Moreover Taylor expansion leads to

$$G^{-1}(i_n/n)=t_n-\xi\frac{1}{n}a_n\sqrt{\log n}\frac{1}{\sqrt{2}\pi\mu_{p,n}(t_n)}+o(1).$$

Hence

$$\sqrt{2}\pi\mu_{p,n}(t_n)\frac{t_n-G^{-1}(i_n/n)}{a_n\frac{\sqrt{\log n}}{n}}=\xi+o(1)$$

and we established the result. $\square$

Real covariance matrices can be considered as well. The first step would be to establish our conclusions for the LOE, the Laguerre Orthogonal Ensemble. This can be done applying *interlacing formulas*, that relates the eigenvalues of LUE and LOE matrices. Forrester and Rains proved in [13] the following relation: $\mathrm{LUE}_{p,n} = \mathrm{even}\big(\mathrm{LOE}_{p,n} \cup \mathrm{LOE}_{p+1,n+1}\big)$. Now we can conclude to similar MDPs for counting functions of eigenvalues in LOE with respect to intervals in the bulk as well as for individual eigenvalues in LOE in the bulk. On the basis of the Four Moment Theorem in the real case, the conclusions can be extended to non-Gaussian real covariance matrices. We omit the details.

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